THERMAL ENTRANCE REGION UNDER HIGHLY NON-ISOVISCOUS FLOW*

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The problem has been solved of simultaneous heat and momentum transfer in an entrance region of a heat exchanger for liquids with highly temperature-dependent viscosity. The method of thermal boundary layer has been applied using a linear approximation of the temperature and a velocity profile corresponding precisely to the temperature. A solution to the boundary layer equation has been outlined and explicit expressions introduced for heat transfer and pressure drop valid for short exchangers with a thin boundary layer.

Heat transfer under forced convection into a liquid flowing in a tube of different wall temperature may be usually described by the equation

$$\varrho c_p v_x \frac{\partial T}{\partial x} = k \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right], \tag{1}$$

with the boundary conditions: $T = T_0$ for $x = 0$, and $T = T_W$ for $r = R$ and $x > 0$. (2), (3)

Even in cases when v_x depends both on r and the axial coordinate x one may often neglect the effect of the radial velocity component and accept Eq. (1). Solution of this equation mandates some numerical method, mostly that of finite differences.

For short exchangers (where $x < 0{\cdot}1 R^2 \varrho c_p U/k$) it is convenient to utilize the method of thermal boundary layer based on assumption that the resistance to heat transfer occurs only in a thin layer adherring to the wall. The set of Eqs (1)–(3) may then be supplemented by another condition, namely

$$T = T_0 \quad \text{and} \quad \partial T/\partial r = 0 \quad \text{for} \quad r = R - \delta\,, \tag{4}$$

permitting a simplified solution to Eq. (1); the function $\delta(x)$ is calculated afterwards. The results obtained by exact solution of the thermal boundary layer equation (provided that the temperature profile in the boundary layer is given by the first eigenfunction of an equation obtained from Eq. (1) (ref.[1,2]) by separation) are in an excellent agreement with the numerical results of the complete set of Eqs (1)–(3) both for isoviscous[3] and non-isoviscous[4] flow. The given approach however, requires numerical approach even to the boundary layer although substantially simpler[1,2] than that for the complete set. A principal advantage of the boundary layer method should be the possibility of obtaining by simple mathematical means an approximate, yet reliable solution.

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Such a solution of non-isoviscous heat transfer by the boundary layer method has been attempted by Murakami[5] who used a very close approximation of the temperature profile (third order power expression) but his superimposed velocity profile resulted in obviously unsound conclusions. The submitted paper though utilizing a rough assumption about the temperature profile (linear profile within the boundary layer) uses a velocity profile that corresponds precisely to the viscosity and shear stress distribution. According to our experience with isoviscous heat transfer this assumption provides a reliable fit to the conditions in short exchangers; the systematic error amounts to about 10%. The proposed method enables to solve simultaneously the problem of coupled heat and momentum transfer analytically with an explicit expression of the results.

A Solution of the Velocity and Temperature Profile in the Boundary Layer

The course of the shear stress across the tube is given by a linear expression as

$$\tau = \tau_s r/R\,, \tag{5}$$

where τ_s is the shear stress on the wall

$$\tau_s = \tfrac{1}{2}(\mathrm{d}p/\mathrm{d}x)\,R\,. \tag{6}$$

The velocity gradient in a Newtonian liquid is

$$\mathrm{d}v_x/\mathrm{d}r = -\,\tau/\mu(r)\,. \tag{7}$$

In accord with the concept of the thermal boundary layer of thickness δ with the temperature varying linearly between T_W and T_0

$$T = T_W + (T_0 - T_W)(R - r)/\delta \quad \text{for} \quad r \geqq R - \delta \tag{8}$$

$$T = T_0 \quad \text{for} \quad r \leqq R - \delta\,, \tag{9}$$

the course of viscosity in the boundary layer is given by

$$\mu = \mu_W \exp\left[\psi(R - r)/\delta\right]. \tag{10}$$

A criterion of isoviscousness, ψ, is determined from the temperature dependence of the viscosity expressed in the investigated range with the aid of an empirical formula

$$\mu = \mu_0 \exp(-\,AT)\,. \tag{11}$$

The dimensionless variables are introduced by

$$w = v_x/u\,, \quad b = (\mathrm{d}p/\mathrm{d}x)\,R^2/(8\mu_W U)\,, \quad y = (R - r)/R\,, \quad y_0 = \delta/R\,. \tag{12)–(15}$$

Combining the definitions (12)–(15) with Eqs (5)–(10) one obtains two differential equations for the velocity profile. For their solution one uses the "no-slip" condition on the wall

$$w = 0 \quad \text{for} \quad y = 0 \tag{16}$$

and the condition of constant flow rate

$$\int_0^1 w(1 - y)\,\mathrm{d}y = \tfrac{1}{2}\,. \tag{17}$$

On defining the auxiliary variables u_1 and u_2 as

$$u_1(y) = w(y)/(4b) \quad \text{for} \quad y \leqq y_0\,, \quad u_2(y) = w(y)/(4b) \quad \text{for} \quad y \geqq y_0\,. \qquad (18), (19)$$

the solution of the differential equations may be expressed in terms of these variables as

$$u_1(y) = \int_0^y (1 - y) \exp(\psi y/y_0)\,\mathrm{d}y\,, \quad u_2(y) = u_1(y_0) + \int_{y_0}^y (1 - y) \exp(-\psi)\,\mathrm{d}y\,. \qquad (20), (21)$$

Denoting further by I_1, I_2, I_3 the following definite integrals

$$I_1 = \int_0^{y_0} u_1(y)\, y/y_0\, (1 - y)\,\mathrm{d}y\,, \tag{22}$$

$$I_2 = \int_{y_0}^1 u_2(y)\,(1 - y)\,\mathrm{d}y\,, \tag{23}$$

$$I_3 = \int_0^{y_0} u_1(y)\,(1 - y)\,\mathrm{d}y\,. \tag{24}$$

the velocity, pressure drop and cup-mixing temperature can be expressed as

$$w(y) = u_1(y)/(I_2 + I_3) \quad \text{for} \quad y \leqq y_0\,, \tag{25}$$

$$w(y) = u_2(y)/(I_2 + I_3) \quad \text{for} \quad y \geqq y_0\,, \tag{26}$$

$$b = 1/[8(I_2 + I_3)]\,, \tag{27}$$

$$t_M = (I_1 + I_2)/(I_2 + I_3)\,. \tag{28}$$

In the calculation of the thickness of the thermal boundary layer, y_0, as a function of y we use the integral balance on heat, which for the given temperature gradient in the boundary layer may be written in dimensionless form as

$$\mathrm{d}t_M/\mathrm{d}z = -2/y_0\,, \tag{29}$$

from which

$$z(y_0) = \frac{1}{2}\int_0^{y_0} y_0(-\,\mathrm{d}t_M/\mathrm{d}y_0)\,\mathrm{d}y_0\,. \tag{30}$$

Thus it suffices to differentiate the function $t_M(y_0)$ determined from Eq. (*29*) and the result is obtained by inversion of the function calculated by integrating Eq. (*30*).

Asymptotic Solution for Low z

Each integration in Eqs (*20*)–(*24*) may be carried out analytically. Moreover, if one neglects in the limit $y_0 \to 0$ higher powers of y_0 the expressions in Eqs (*25*)–(*28*) may be given by particularly simple functions of y_0 and ψ. Thus then dimensionless cup-mixing temperature is

$$t_M = 1 - \tfrac{4}{3}y_0^2\,[1 - f(\psi)]\,, \tag{31}$$

where

$$f(\psi) = 1 + 6\left[\frac{1}{\psi^3} - \exp\psi\left(\frac{1}{\psi^3} + \frac{1}{2\psi} - \frac{1}{\psi^2}\right)\right]. \tag{32}$$

On substituting Eq. (*31*) into Eq. (*30*), integrating and inverting we get

$$y_0 = \left[\frac{9z}{4[1 - f(\psi)]}\right]^{1/3}. \tag{33}$$

The last equation permits the variables calculated for $y_0 \to 0$ from Eqs (*27*) and (*28*) to be expressed as functions of z:

$$t_M = 1 - \tfrac{4}{3}(9/4)^{2/3}\,z^{2/3}[1 - f(\psi)]^{1/3}\,, \tag{34}$$

$$b = \exp\psi\{1 - z^{1/3}\,4(9/4)^{1/3}[1 + (1/\psi) - (\exp\psi/\psi)]/[1 - f(\psi)]^{1/3}\}^{-1}\,. \tag{35}$$

An interesting result is the value of the coefficient α defined as

$$\mathrm{d}\ln(\mathrm{Nu}_{\mathrm{noniso}}/\mathrm{Nu}_{\mathrm{iso}})/\mathrm{d}\ln\left(\frac{\mu_W}{\mu_M}\right) = -\ln[1 - f(\psi)]/(3\psi)\,. \tag{36}$$

The course of this coefficient is plotted in Fig. 3. As it may be seen α does not depend strongly on ψ and hence one can accept the following relation

$$\mathrm{Nu}_{\mathrm{noniso}} = \mathrm{Nu}_{\mathrm{iso}}(\mu_{\mathrm{W}}/\mu_{\mathrm{M}})^{0 \cdot 25}\,, \tag{37}$$

where 0·25 is the value of α at $\psi \to 0$. The high value of the exponent (0·25) instead

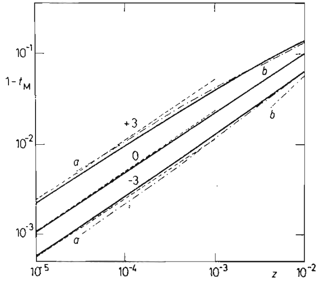

FIG. 1

Dimensionless Cup-Mixing Temperature under Non-Isoviscous Flow in Short Exchangers

——— Solution by thermal boundary layer method[1,2], — — — approximative solution for low z (Eq. (*34*)), —.—.—.— numerical solution by finite difference method a due to Christiansen, Jensen and Tao[4], axial increment about 10^{-6}, b due to Kwant[9], axial increment $4 \cdot 10^{-4}$. Figures on curves indicate values of ψ.

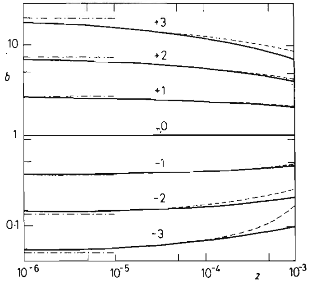

FIG. 2

Dimensionless Local Pressure Drop as a Function of z

——— Solution by boundary layer method[1,2] — — — approximative solution for low z (Eq. (*35*)), —.—.—.— pressure drop corresponding to inlet viscosity. Figures on curves indicate values of ψ.

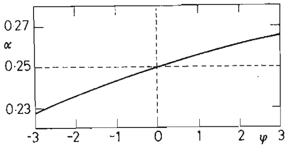

FIG. 3

Calculated Exponent of the Sieder–Tate correction as a Function of ψ for $z \to 0$.

of the more usual $\alpha = 0.14$ appearing in the correlation of the Sieder-Tate type[6] (Eq. (*37*)) will be suitable just for the exchangers with short time of contact (small z). This is *e.g.* the case of mixed vessels with laminar regime near the wall (this occurs even for fairly high values of the Reynolds number for mixing), where, as mentioned by Uhl[7], the experimentally found values of the exponent were $\alpha = 0.24$ and $\alpha = 0.25$. The often used value $\alpha = 0.14$ stems from experiments of Sieder and Tate for long tubular exchangers[6].

DISCUSSION

1) The validity of the above solution may be judged for instance from comparison of Eq. (*34*) for isoviscous conditions:

$$t_M = 1 - 2.29z^{2/3} \tag{34a}$$

with the exact solution due to Levéque[8]

$$t_M = 1 - 2.56z^{2/3} \tag{38}$$

The relative error dt_M/dz amounts to 10·5%.

2) The agreement with the solution of non-isoviscous heat transfer may be judged from Figs 1 and 2 plotting the results obtained by solving the boundary layer equation under more realistic assumptions[1,2]. As it is apparent the results of the complete set Eqs (*1*)–(*3*) and (*5*)–(*7*)[4,9] obtained by a numerical technique comport well with those of the presented method for higher values of z. At low values of z (where y_0 becomes comparable with the increment of the numerical computational grid) the numerical results lose their accuracy, while Eqs (*31*)–(*37*) hold in this very region.

3) The whole procedure can be repeated for non-Newtonian liquids, whose non-isoviscous flow is governed by

$$dv_x/dr = -(\tau/\eta_0)^{1/n} \exp(AT), \tag{39}$$

The procedure becomes particularly simple if $1/n$ is an integer[10].
The result is

$$t_M = 1 - 2.29\left(\frac{3n+1}{4n}\right)^{1/3} z^{2/3}\,[1 - f(\psi)]^{1/3}, \tag{40}$$

where the correction of non-Newtonian behaviour is identical to that derived by Pigford[11]. Furthermore, this case confirms theoretically the appropriateness of the definition of the parameter A for non-Newtonian liquids as

$$A = \left.\frac{\partial \ln \eta}{\partial t}\right|_{\tau=\mathrm{const.}} \quad (41)$$

since it enables the effects of non-Newtonian behaviour and that of non-isoviscousness on heat transfer to be separated.

4) The above presented method is suitable, as it is apparent from Figs 1 and 2, for exchangers satisfying $z \ll 0{\cdot}001$. The other stipulation is that $z \gg \mathrm{Pe}^{-3/2}$ for we could neglect the effect of axial conduction on the over-all heat transfer[12]. Under low values of Pe (*e.g.* for slow motion of liquid metals in capillaries) even the boundary conditions in Eqs (*2*) and (*3*) are difficult to realize. From the continuity equation and the velocity profiles (*25*) it further follows that for exchangers having $z \ll |\psi|\, \mathrm{Pe}^{-3/2}$ one has to take into consideration also the effects of radial velocity and inertia forces.

Nevertheless, the range $\mathrm{Pe}^{-3/2} \ll z \ll 0{\cdot}001$ into which the presented results are confined covers a broad class of exchangers of practical interest; as has been shown some of the results hold not only for flow in tubes but also for agitated vessels used for heat transfer operations.

LIST OF SYMBOLS

A	material constant, Eq. (*11*), (*41*), (deg^{-1})
b	dimensionless pressure drop
c_p	heat capacity, ($\mathrm{cal\ g^{-1}\ deg^{-1}}$)
f	function defined by Eq. (*32*)
I_1, I_2, I_3	integrals defined by Eqs (*22*)–(*24*)
k	thermal conductivity, ($\mathrm{cal\ cm^{-1}\ s^{-1}\ deg^{-1}}$)
n	flow index
$\mathrm{Nu}_{\mathrm{noniso}}$	average Nusselt number based on arithmetic average temperature difference
$\mathrm{Nu}_{\mathrm{iso}}$	Nusselt number for isoviscous flow
$\mathrm{d}p/\mathrm{d}x$	local pressure drop, ($\mathrm{g\ cm^{-3}\ s^{-2}}$)
$\mathrm{Pe} = 2c_pUR/k$	Peclet number
r	radial coordinate, (cm)
R	radius of tube, (cm)
T	temperature, (deg C)
T_0	inlet temperature of liquid, (deg C)
T_W	wall temperature, (deg C)
$t_M = (T_M - T_W)/(T_0 - T_W)$	dimensionlesss cup-mixing temperature
T_M	cup-mixing temperature, (deg C)
U	average velocity, ($\mathrm{cm\ s^{-1}}$)
u_1, u_2	auxiliary functions defines by Eqs. (*18*), (*19*)
v_x	axial component of velocity, ($\mathrm{cm\ s^{-1}}$)
w	dimensionless velocity

x axial coordinate of examined cross section, (cm)
y dimensionless distance from wall
y_0 dimensionless thickness of boundary layer
$z = 2x/(R\,\mathrm{Pe})$ dimensionless axial coordinate
α exponent of Sider-Tate correction, Eq. (*36*)
δ thickness of thermal boundary layer, (cm)
η apparent viscosity of non-Newtonian liquid (g cm^{-1} s^{-1})
η_0 material constant, (g cm^{-1} s^{2-n})
μ viscosity, (g cm^{-1} s^{-1})
μ_0 material constant, (g cm^{-1} s^{-1})
μ_M viscosity at temperature T_M, (g cm^{-1} s^{-1})
μ_W viscosity at temperature T_W, (g cm^{-1} s^{-1})
ϱ density, (g cm^{-3})
τ shear stress, (g cm^{-1} s^{-2})
τ_W wall shear stress, (g cm^{-1} s^{-})
$\psi = A(T_W - T_0)$ criterion of non-isoviscousness

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